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Lie symmetries of a coupled nonlinear Burgers–heat equation system

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Abstract. A symmetry group analysis of a coupled Burgers–heat equation system of partial differential equations initially introduced in a study of non-classical similarity solutions of the heat equation is carried out. The point Lie algebra of the system $\mathscr{G}$, is shown to possess a maximal solvable ideal $\mathscr{A}$ with quotient algebra $\mathscr{B}=\mathscr{G}/\mathscr{A}\simeq \mathrm{sl}_2(R)$, where $\mathrm{sl}_2(R)$ is the Lie algebra of 2×2 matrices with zero trace. Analysis of the prolongation structure of the system yields the Bäcklund transformation obtained previously by Painlevé analysis. The Bäcklund transformation can be expressed as a map onto two coupled linear heat equations, or alternatively as a map onto Burgers equation. The role of $\mathrm{sl}_2(R)$ in a seven-dimensional Lie algebra, $\mathscr{L}_1$, obtained by truncating the open-ended algebraic prolongation structure is emphasised.

1. Introduction

Lie group theoretical methods of solving partial differential equations have been developed by a number of authors (e.g. Bluman and Cole [1, 2], Harrison and Estabrook [3], Wahlquist and Estabrook [4], Ovsjannikov [5], Olver [6] and Kaup [7]).

In their work on non-classical similarity solutions of the heat equation

$$\Psi_{xx}-\Psi_t=0 \tag{1.1}$$

Bluman and Cole [1] and Harrison and Estabrook [3] considered the associated nonlinear system

$$A_t+2AA_x-A_{xx}+2C_x=0 \tag{1.2}$$

$$C_t-C_{xx}+2CA_x=0 \tag{1.3}$$

$$D_t-D_{xx}+2DA_x=0. \tag{1.4}$$

Integration of the characteristic equations $\mathrm{d}x/\mathrm{d}t=A(x,t)$, and $\mathrm{d}\Psi/\mathrm{d}t=C(x,t)\Psi+D(x,t)$ then leads to the generation of similarity solutions of the heat equation. 'Classical' similarity solutions of the heat equation correspond to solutions of the system (1.2)-(1.4) for which $A_{xx}=0$. Kaup [7] has considered the prolongation structure for both Burgers equation and a 'damped Burgers equation'. By exploiting an eigenvalue in the associated linear scattering equations (obtained by prolongation) Kaup was able to devise an inverse scattering scheme to solve the damped Burgers equation. Equations (1.2)-(1.4), although related to Burgers equation are different from the equations considered by Kaup.

Painlevé analysis of the nonlinear equations (1.2) and (1.3) by Webb [8] showed that the system (1.2)-(1.3) has two singularity branches of interest. One of the singularity branches yielded a Bäcklund transformation that mapped the nonlinear system (1.2) and (1.3) onto two coupled linear heat equations. The second singularity branch also yielded solutions of interest. These solutions for $A(x, t)$ and $C(x, t)$ were used to generate examples of non-classical similarity solutions of the heat equation.

In the present paper, we consider some of the group theoretical properties of the system (1.2)-(1.3). In section 2 we obtain the Lie group of the system (1.2)-(1.3), and use this to delineate the form of the classical similarity solutions of equations (1.2) and (1.3). The prolongation structure and associated Lie algebra for the system (1.2)-(1.3) is then obtained (section 3) by the methods of Wahlquist and Estabrook [4]. The Bäcklund transformation previously obtained by Webb [8] by Painlevé analysis is obtained from the prolongation structure. Section 4 concludes with a summary and discussion.

2. Point Lie group and classical similarity solutions

To obtain the Lie group of equations (1.2)-(1.3) we use the exterior differential forms approach of Harrison and Estabrook [3]. The system (1.2)-(1.3) can be represented by the set of forms

$$\alpha_1 = \mathrm{d}x \wedge \mathrm{d}A + 2A\, \mathrm{d}A \wedge \mathrm{d}t - \mathrm{d}p \wedge \mathrm{d}t + 2\, \mathrm{d}C \wedge \mathrm{d}t \tag{2.1}$$

$$\alpha_2 = \mathrm{d}x \wedge \mathrm{d}C - \mathrm{d}q \wedge \mathrm{d}t + 2C\, \mathrm{d}A \wedge \mathrm{d}t \tag{2.2}$$

$$\alpha_3 = p\, \mathrm{d}x \wedge \mathrm{d}t - \mathrm{d}A \wedge \mathrm{d}t \tag{2.3}$$

$$\alpha_4 = q\, \mathrm{d}x \wedge \mathrm{d}t - \mathrm{d}C \wedge \mathrm{d}t \tag{2.4}$$

where '$\wedge$' denotes the exterior product and, 'd' denotes exterior differentiation. Sectioning the forms (2.1)-(2.4) by making all variables depend only on x and t yields

$$A_t + 2AA_x - p_x + 2C_x = 0 \tag{2.5}$$

$$C_t - q_x + 2CA_x = 0 \tag{2.6}$$

$$p = A_x \qquad q = C_x. \tag{2.7}$$

In addition the forms $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ form a closed ideal (i.e. the exterior derivatives of the forms, $\mathrm{d}\alpha_i$, are also in the ideal).

To determine the infinitesimal generators $V = (V^x, V^t, V^A, V^C, V^p, V^q)$ of the Lie group admitted by the system we require that the Lie derivative of each of the α_i $(i = 1, 2, 3, 4)$ also be in the ideal, i.e.,

$$\mathscr{L}_V(\alpha_i) = \sum_{j=1}^{4} a_{ij}\alpha_j \qquad i, j = 1, 2, 3, 4 \tag{2.8}$$

where the a_{ij} are scalars. Solving the determining equations (2.8) for V by the standard method (e.g. Harrison and Estabrook [3]) we find

$$V^t = a_1 + 2a_2t + 2a_3t^2 \tag{2.9}$$

$$V^x = 2a_3xt + a_2x + 2a_5t + a_4 \tag{2.10}$$

$$V^A = -(2a_3t + a_2)A + 2a_3x + 2a_5 \tag{2.11}$$

$$V^C = -(4a_3t + 2a_2)C - (a_5 + a_3x)A - a_3 \tag{2.12}$$

$$V^p = -p(4a_3t + 2a_2) + 2a_3 \tag{2.13}$$

$$V^q = -q(6a_3t + 3a_2) - (a_5 + a_3x)p - a_3A \tag{2.14}$$

where the a_i, $i = 1, 2, 3, 4, 5$ are constants.

A basis $\{X_i, 1 \leq i \leq 5\}$ for the Lie algebra $\mathscr{G}$ associated with the infinitesimal generators V of the invariance group G can be obtained by setting $a_i = 1$ for a fixed i, and $a_j = 0$, $j \neq i$. We find

$$X_1 = \partial_t \tag{2.16}$$

$$X_2 = 2t\partial_t + x\partial_x - A\partial_A - 2C\partial_C - 2p\partial_p - 3q\partial_q \tag{2.17}$$

$$\begin{aligned} X_3 = {} & 2t^2\partial_t + 2xt\partial_x + (2x - 2tA)\partial_A + (-4tC - xA - 1)\partial_C \\ & + (2 - 4pt)\partial_p + (-6qt - xp - A)\partial_q \end{aligned} \tag{2.18}$$

$$X_4 = \partial_x \tag{2.19}$$

$$X_5 = 2t\partial_x + 2\partial_A - A\partial_C - p\partial_q \tag{2.20}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_x \equiv \partial/\partial x$ etc. The commutators are given in table 1.

The algebra $\mathscr{G}$ has a maximal solvable ideal $\mathscr{A}$ spanned by $\{X_4, X_5\}$ (i.e. $\mathscr{A}' = [\mathscr{A}, \mathscr{A}] = 0$ (Jacobson [9], p 24)). The quotient algebra $\mathscr{B} = \mathscr{G}/\mathscr{A}$ is spanned by the additive cosets $\tilde{X}_i = X_i + \mathscr{A}$, $1 \leq i \leq 3$ and the Levi decomposition of $\mathscr{G}$ gives

$$\mathscr{G} \simeq \mathscr{B} \oplus \mathscr{A}. \tag{2.21}$$

The quotient algebra $\mathscr{B}$ is isomorphic to $\mathrm{sl}_2(R)$, the algebra of real 2×2 matrices with zero trace. To see this we note that

$$Z_1 = \tfrac{1}{2}X_1 \qquad Z_2 = X_2 \qquad Z_3 = -X_3 \tag{2.22}$$

forms a basis for the sub-algebra $\{X_1, X_2, X_3\}$ with

$$[Z_2, Z_3] = 2Z_3 \qquad [Z_2, Z_1] = -2Z_1 \qquad [Z_3, Z_1] = Z_2 \tag{2.23}$$

and $\tilde{Z}_i = \{Z_i + \mathscr{A}, 1 \leq i \leq 3\}$ spans $\mathscr{B}$. On the other hand, the matrices

$$W_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad W_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad W_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.24}$$

have commutation relations

$$[W_2, W_3] = 2W_3 \qquad [W_2, W_1] = -2W_1 \qquad [W_3, W_1] = W_2 \tag{2.25}$$

and $\{W_1, W_2, W_3\}$ span $\mathrm{sl}_2(R)$. From equations (2.23) and (2.25) we find $\mathscr{B} \simeq \mathrm{sl}_2(R)$. These results are similar to those obtained by Sastri and Dunn [10], who show that the quotient algebra of the heat equation is isomorphic to $\mathrm{sl}_2(R)$.

Table 1. Commutators for the point Lie algebra.

	X_1	X_2	X_3	X_4	X_5
X_1	0	$2X_1$	$2X_2$	0	$2X_4$
X_2	$-2X_1$	0	$2X_3$	$-X_4$	X_5
X_3	$-2X_2$	$-2X_3$	0	$-X_5$	0
X_4	0	X_4	X_5	0	0
X_5	$-2X_4$	$-X_5$	0	0	0

We conclude this section by providing a similarity reduction of the original equations (1.2)-(1.3). Integrating the characteristic equations

$$\frac{\mathrm{d}x}{V^x}=\frac{\mathrm{d}t}{V^t}=\frac{\mathrm{d}A}{V^A}=\frac{\mathrm{d}C}{V^C} \tag{2.26}$$

we obtain similarity solutions for A and C of the form (cf Bluman and Cole [1, 2]):

$$A=[2a_3\eta t+\alpha\theta^{1/2}+G(\eta)]\theta^{-1/2} \tag{2.27}$$

$$C=\theta^{-1}\left\{-a_3t+H(\eta)-\left(\frac{\alpha}{2}\theta^{1/2}+a_3\eta t\right)G(\eta)-a_3\alpha\eta t\theta^{1/2}-\tfrac{1}{2}a_2\alpha^2t-\tfrac{1}{2}a_3\alpha^2t^2-a_3^2\eta^2t^2\right\} \tag{2.28}$$

where

$$\theta=2a_3t^2+2a_2t+a_1 \tag{2.29}$$

$$\eta=(x-\alpha t-\beta)\theta^{-1/2} \tag{2.30}$$

$$\alpha=2(a_3a_4-a_5a_2)/(2a_3a_1-a_2^2) \tag{2.31}$$

$$\beta=(a_4a_2-2a_5a_1)/(2a_3a_1-a_2^2). \tag{2.32}$$

Equation (2.30) gives the similarity variable η, obtained by integrating the equation $\mathrm{d}x/\mathrm{d}t=V^x/V^t$ of equations (2.26). The similarity variable η is the same as that used by Bluman and Cole [1, 2] in obtaining classical similarity solutions of the heat equation. Requiring the solution forms (2.27)-(2.28) for A and C to satisfy the original equations (1.2)-(1.3) yields ordinary differential equations for $G(\eta)$ and $H(\eta)$:

$$G''-2GG'-2H'(\eta)+a_2(\eta G'+G)-2a_1a_3\eta=0 \tag{2.33}$$

$$H''-2HG'-\frac{a_1\alpha^2}{2}G'+a_2\eta H'+a_1a_3\eta G+2a_2H+a_1(a_3+a_2\alpha^2/2)=0. \tag{2.34}$$

Further reduction of the system entails the solution of the ordinary differential equations (2.33) and (2.34).

3. The prolongation structure

Following the approach of Wahlquist and Estabrook [4], we search for 1-forms

$$\omega_k=\mathrm{d}y_k+F^k(z_\mu,y_\nu)\,\mathrm{d}x+G^k(z_\mu,y_\nu)\,\mathrm{d}t \tag{3.1}$$

where $z_\mu=(x,t,A,C,p,q)$. The y_ν are pseudopotentials and the ω_k are such that $\mathrm{d}\omega_k$ is in the prolonged ideal. Thus we require

$$\mathrm{d}\omega_k=f_i^k\alpha_i+\eta_j^k\wedge\omega_j \tag{3.2}$$

where the Einstein summation convention is used for repeated indices and the 2-forms $\{\alpha_i\}$, $i=1,2,3,4$, given by equations (2.1)-(2.4) represent the original partial differential equations (1.2)-(1.3). The η_i^k are 1-forms and the f_i^k are scalars. The condition (3.2) ensures that the prolonged ideal spanned by $\{\alpha_i\}$ and the $\{\omega_j\}$ is closed. The number of ω_j and y_j in equation (3.1) and (3.2) is arbitrary at this point. By the usual identity for the exterior derivative of any differential form $\mathrm{d}\,\mathrm{d}\omega_k=0$, so that the forms (3.2) are exact.

Taking the exterior derivative of equation (3.1) and equating the result for $d\omega_k$ with that in equation (3.2) yields a set of determining equations for the F^k and G^k. We restrict our attention to solutions for which F^k and G^k are independent of x and t (i.e. $F^k = F^k(A, C, p, q, y_j)$, $G^k = G^k(A, C, p, q, y_j)$. We obtain

$$F^k = AX_1^k + CX_2^k + X_3^k \tag{3.3}$$

$$G^k = pX_1^k + qX_2^k - ACX_2^k + (X_5^k - 2X_1^k)C - X_1^k A^2 + X_6^k A + X_4^k \tag{3.4}$$

where the vector fields

$$X_\alpha = X_\alpha^k(y)\partial_k \tag{3.5}$$

($\partial_k = \partial/\partial y_k$) satisfy the commutator equations

$$\begin{aligned} &X_5 = [X_3, X_2] \qquad X_6 = [X_3, X_1] \\ &[X_2, X_1] = X_2 \qquad [X_2, X_5] = 2X_2 \qquad [X_1, X_6] = X_6 \\ &[X_1, X_5] = 0 \qquad [X_3, X_4] = 0 \qquad [X_1, X_4] + [X_3, X_6] = 0 \\ &[X_2, X_4] + [X_3, X_5] = 2X_6. \end{aligned} \tag{3.6}$$

Note that these equations do not depend on the number N of prolongation variables assumed. Equations (3.3)-(3.6) can in fact be written in terms of X_1, X_2, X_3 and X_4 with the first two equations (3.6) simply defining X_5 and X_6.

For the X_α to form a Lie algebra we require that the Jacobi identity apply. Using the Jacobi identity in equations (3.6) we obtain

$$\begin{aligned} &[X_2, X_6] = X_5 \qquad [X_5, X_6] = 2X_6 \\ &[X_2, [X_1, X_4]] = 0 \qquad [X_1, [X_2, X_4]] = 0. \end{aligned} \tag{3.7}$$

Defining

$$X_7 = [X_1, X_4] \qquad X_8 = [X_2, X_4] \tag{3.8}$$

allows the last two equations of (3.6) to be written as

$$[X_3, X_6] = -X_7 \qquad [X_3, X_5] = 2X_6 - X_8 \tag{3.9}$$

whereas the last two equations of (3.7) read

$$[X_2, X_7] = 0 \qquad [X_1, X_8] = 0. \tag{3.10}$$

Further operations will give further commutation relations, but does not lead to a closed system implying an open structure with an infinite number of prolongation variables. A detailed investigation of this infinite algebraic prolongation structure is of interest but will not be pursued further here.

In the present paper we simply impose closure on the system by requiring that X_8 be linearly dependent on the $\{X_i, 1 \leqslant i \leqslant 7\}$. We find

$$X_8 = -\alpha_1(X_5 - 2X_1) + \alpha_4(X_4 - X_7) - (1 + \alpha_1\alpha_4)(X_3 + X_6) \tag{3.11}$$

where α_1 and α_4 are arbitrary constants. The commutators for the $\{X_i\}$ are given in table 2. Note that this Lie algebra is seven-dimensional since X_8 depends linearly on the $\{X_i, 1 \leqslant i \leqslant 7\}$.

We now set

$$Y_i = X_i \qquad i = 1, 2, 3, 4, 7, 8$$
$$Y_5 = X_5 - 2X_1 \qquad Y_6 = X_6 + X_3 \tag{3.12}$$

Table 2. Commutators for the prolongation space Lie algebra.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_2$	$-X_6$	X_7	0	X_6	X_7	0
X_2	X_2	0	$-X_5$	X_8	$2X_2$	X_5	X_8	0
X_3	X_6	X_5	0	0	$2X_6-X_8$	$-X_7$	$\alpha_1 X_7$	$\alpha_1 X_8+X_7$
X_4	$-X_7$	$-X_8$	0	0	$-\alpha_1 X_8-X_7$	$-\alpha_1 X_7$	$\alpha_1^2 X_7$	$\alpha_1^2 X_8$
X_5	0	$-2X_2$	X_8-2X_6	$\alpha_1 X_8+X_7$	0	$2X_6$	X_7	$-X_8$
X_6	$-X_6$	$-X_5$	X_7	$\alpha_1 X_7$	$-2X_6$	0	0	$-X_7$
X_7	$-X_7$	$-X_8$	$-\alpha_1 X_7$	$-\alpha_1^2 X_7$	$-X_7$	0	0	0
X_8	0	0	$-\alpha_1 X_8-X_7$	$-\alpha_1^2 X_8$	X_8	X_7	0	0

corresponding to a change of basis vectors for the Lie algebra of table 2. The commutators for the $\{Y_i\}$ are given in table 3. In this form we see that the truncated Lie algebra $\mathscr{L}_1$ has a solvable ideal $\mathscr{A}_1 = \{Y_4, Y_5, Y_6, Y_7\}$. The first derived ideal $\mathscr{A}_1' = [\mathscr{A}_1, \mathscr{A}_1] = \{Y_7, Y_8\}$, whereas $\mathscr{A}_1'' = [\mathscr{A}_1', \mathscr{A}_1'] = \{0\}$ so that $\mathscr{A}_1$ is solvable. The quotient algebra $\mathscr{B}_1 = \mathscr{L}_1/\mathscr{A}_1$ is spanned by the cosets $\tilde{Z}_1 = -Y_2 + \mathscr{A}_1$, $\tilde{Z}_2 = 2Y_1 + \mathscr{A}_1$, $\tilde{Z}_3 = -Y_3 + \mathscr{A}_1$, with commutation relations

$$[\tilde{Z}_2, \tilde{Z}_3] = 2\tilde{Z}_3 \qquad [\tilde{Z}_2, \tilde{Z}_1] = -2\tilde{Z}_1 \qquad [\tilde{Z}_3, \tilde{Z}_1] = \tilde{Z}_2.$$

By using the map $\tilde{Z}_i \to W_i$ $(i = 1, 2, 3)$ where $\{W_1, W_2, W_3\}$ are the 2×2 matrices (2.24) that span $\mathrm{sl}_2(R)$ we see that $\mathscr{B}_1 \simeq \mathrm{sl}_2(R)$. The centre of the algebra $\mathscr{L}_1 = \{0\}$. Thus not only is the quotient algebra of the point Lie group, $\mathscr{B} = \mathscr{G}/\mathscr{A} \simeq \mathrm{sl}_2(R)$ (equations 2.21–2.24) but the quotient Lie algebra in the prolongation space $\mathscr{B}_1 = \mathscr{L}_1/\mathscr{A}_1$ is also isomorphic to $\mathrm{sl}_2(R)$.

A realisation of the Lie algebra of table 3 is spanned by the basis vectors

$$Y_\alpha = C^{\mu}_{\alpha\nu} y_\mu \frac{\partial}{\partial y_\nu} \tag{3.13}$$

where $C^{\mu}_{\alpha\nu}$ are the structure constants of the algebra (i.e., $[Y_\alpha, Y_\beta] = C^{e}_{\alpha\beta} Y_e$), which can be read off table 3. Using the realisation (3.13) for the Y_α, and equations (3.1)–(3.4)

Table 3. Commutators for the $\{Yi, 1 \leqslant i \leqslant 8\}$.

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Y_1	0	$-Y_2$	Y_3-Y_6	Y_7	0	0	Y_7	0
Y_2	Y_2	0	$-2Y_1-Y_5$	Y_8	0	0	Y_8	0
Y_3	Y_6-Y_3	$2Y_1+Y_5$	0	0	$-Y_8$	$-Y_7$	$\alpha_1 Y_7$	$\alpha_1 Y_8+Y_7$
Y_4	$-Y_7$	$-Y_8$	0	0	$-\alpha_1 Y_8+Y_7$	$-\alpha_1 Y_7$	$\alpha_1^2 Y_7$	$\alpha_1^2 Y_8$
Y_5	0	0	Y_8	$\alpha_1 Y_8-Y_7$	0	Y_8	$-Y_7$	$-Y_8$
Y_6	0	0	Y_7	$\alpha_1 Y_7$	$-Y_8$	0	$\alpha_1 Y_7$	$\alpha_1 Y_8$
Y_7	$-Y_7$	$-Y_8$	$-\alpha_1 Y_7$	$-\alpha_1^2 Y_7$	Y_7	$-\alpha_1 Y_7$	0	0
Y_8	0	0	$-\alpha_1 Y_8-Y_7$	$-\alpha_1^2 Y_8$	Y_8	$-\alpha_1 Y_8$	0	0

and (3.12) we obtain the Pfaffians

$$
\begin{aligned}
\omega_1 &= dy_1+(Cy_2+y_6-y_3)\,dx+[(q-AC)y_2+(y_3-y_6)A-y_7]\,dt \\
\omega_2 &= dy_2+(-Ay_2+2y_1+y_5)\,dx+[(A^2-p)y_2-(2y_1+y_5)A-y_8]\,dt \\
\omega_3 &= dy_3+[A(y_3-y_6)-C(2y_1+y_5)]\,dx+[(p-A^2)(y_3-y_6) \\
&\qquad +(AC-q)(y_5+2y_1)+Ay_7+Cy_8]\,dt \\
\omega_4 &= dy_4+[Ay_7+Cy_8]\,dx+[(p-A^2-C+\alpha_1 A)y_7+(q-AC+\alpha_1 C)y_8]\,dt \\
\omega_5 &= dy_5-y_8\,dx+(-\alpha_1 y_8+y_7)\,dt \\
\omega_6 &= dy_6-y_7\,dx+[Cy_8+(A-\alpha_1)y_7]\,dt \\
\omega_7 &= dy_7+[(A+\alpha_1)y_7+Cy_8]\,dx+[(p-C-A^2+\alpha_1^2)y_7+(q-AC)y_8]\,dt \\
\omega_8 &= dy_8+(\alpha_1 y_8+y_7)\,dx+[-Ay_7+(\alpha_1^2-C)y_8]\,dt
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
y_8 &= -\alpha_1 y_5-(1+\alpha_1\alpha_4)y_6+\alpha_4(y_4-y_7) \\
\omega_8 &= -\alpha_1\omega_5-(1+\alpha_1\alpha_4)\omega_6+\alpha_4(\omega_4-\omega_7).
\end{aligned}
\tag{3.15}
$$

As an example of the use of these Pfaffians consider the first order differential system obtained by sectioning the forms ω_7 and ω_8 (i.e. set $y_7=y_7(x,t)$, $y_8=y_8(x,t)$ and put $\omega_7=\omega_8=0$). We obtain

$$
\frac{\partial}{\partial x}\begin{bmatrix} y_7 \\ y_8 \end{bmatrix}=\begin{bmatrix} -(A+\alpha_1) & -C \\ -1 & -\alpha_1 \end{bmatrix}\begin{bmatrix} y_7 \\ y_8 \end{bmatrix} \tag{3.16}
$$

$$
\frac{\partial}{\partial t}\begin{bmatrix} y_7 \\ y_8 \end{bmatrix}=\begin{bmatrix} (A^2+C-A_x-\alpha_1^2) & (AC-C_x) \\ A & (C-\alpha_1^2) \end{bmatrix}\begin{bmatrix} y_7 \\ y_8 \end{bmatrix}. \tag{3.17}
$$

The integrability conditions for equations (3.16) and (3.17) (i.e. $y_{7,xt}=y_{7,tx}$ and $y_{8,xt}=y_{8,tx}$) are simply equations (1.2) and (1.3) for A and C. Equations (3.16) and (3.17) are similar to the linear scattering equations of the AKNS scheme (e.g. Ablowitz *et al* [11], Ablowitz and Segur [12], Newell [13]). The parameter α_1 might at first glance be regarded as an eigenvalue. However, the transformations

$$
\tilde{y}_7=y_7\exp(\alpha_1 x+\alpha_1^2 t) \qquad \tilde{y}_8=y_8\exp(\alpha_1 x+\alpha_1^2 t) \tag{3.18}
$$

when substituted in equations (3.16) and (3.17) yield equations of the form (3.16) and (3.17) for $\tilde{y}_7$ and $\tilde{y}_8$, but with $\alpha_1=0$. Thus α_1 is not an eigenvalue. It also follows from equations (3.16) and (3.17) that y_7 and y_8 satisfy the convection-diffusion equations

$$
y_{7,t}-y_{7,xx}-2\alpha_1 y_{7,x}=0 \tag{3.19}
$$

$$
y_{8,t}-y_{8,xx}-2\alpha_1 y_{8,x}=0 \tag{3.20}
$$

with $(-2\alpha_1)$ corresponding to the convection speed. Sectioning the other forms it is not difficult to show that the potentials y_4 and y_5 also satisfy the convection diffusion equation (3.19).

The linear matrix scheme (3.16)-(3.17) can also be used to obtain Bäcklund transformations. Setting

$$
\gamma=y_7/y_8 \tag{3.21}
$$

we obtain the Riccati equations

$$\gamma_x = \gamma^2 - A\gamma - C \tag{3.22}$$

$$\gamma_t = (AC - C_x) + (A^2 - A_x)\gamma - A\gamma^2. \tag{3.23}$$

The integrability condition $\gamma_{xt} = \gamma_{tx}$ again requires A and C to satisfy equations (1.2) and (1.3). From equations (3.22) and (3.23) we find that γ satisfies Burgers equation

$$\gamma_t + 2\gamma\gamma_x - \gamma_{xx} = 0. \tag{3.24}$$

Hence equations (3.22)-(3.24) constitute a Bäcklund transformation between Burgers equation and the original equations (1.2) and (1.3).

An alternative form of the Bäcklund transformation is obtained by using the Cole-Hopf transformation

$$\gamma = -w_x / w \tag{3.25}$$

in equation (3.24) to obtain

$$w_t - w_{xx} = \mu(t) w \tag{3.26}$$

where μ is an arbitrary function of t. Noting

$$C = \gamma^2 - A\gamma - \gamma_x \tag{3.27}$$

and setting

$$A = -\theta_x / \theta \qquad \theta = w\vartheta \tag{3.28}$$

equation (1.2) becomes

$$\vartheta_t - \vartheta_{xx} - 2\vartheta \frac{\partial^2}{\partial x^2} \ln w = \lambda(t)\vartheta \tag{3.29}$$

where $\lambda(t)$ is an arbitrary function of t. In terms of w and ϑ equations (3.27) and (3.28) yield

$$A = -\frac{\partial}{\partial x} \ln(w\vartheta) \tag{3.30}$$

$$C = \frac{\partial^2}{\partial x^2} \ln w - \frac{\partial \ln w}{\partial x} \frac{\partial \ln \vartheta}{\partial x}. \tag{3.31}$$

Equations (3.26), (3.29), (3.30) and (3.31) are the form of the Bäcklund transformation for equations (1.2) and (1.3) derived in Webb [8] by Painlevé analysis.

A more elegant derivation of the Riccati equations (3.22) and (3.23) can in fact be obtained by noting that if we arbitrarily set $Y_4 = Y_5 = Y_6 = Y_7 = Y_8 = 0$ in table 3 we obtain a closed Lie algebra with basis vectors $\{Y_1, Y_2, Y_3\}$ which is isomorphic to $\mathrm{sl}_2(R)$. Exploiting the known, one-dimensional realisation of $\mathrm{sl}_2(R)$ (e.g. Miller [14]) by the one-dimensional vectors

$$Y_1 = z\frac{\mathrm{d}}{\mathrm{d}z} \qquad Y_2 = \frac{\mathrm{d}}{\mathrm{d}z} \qquad Y_3 = -z^2 \frac{\mathrm{d}}{\mathrm{d}z} \tag{3.32}$$

equations (3.1)-(3.4) and (3.12) yield the Pfaffian

$$\omega_9 = \mathrm{d}z + [C + Az - z^2]\,\mathrm{d}x + [C_x - AC + z(A_x - A^2) + Az^2]\,\mathrm{d}t. \tag{3.33}$$

Sectioning the form (3.33) yields the Riccati equations (3.22) and (3.23) but with the γ replaced by z.

It is worth noting that equations (3.16) and (3.17) combined with equations (3.3) and (3.4) may be used to obtain a 2×2 matrix representation of the Lie algebra of table 2 as

$$\begin{aligned} &X_1=\tfrac{1}{2}(W_2+I) \qquad X_2=W_1 \qquad X_3=\alpha_1 I+W_3 \\ &X_4=\alpha_1^2 I \qquad X_5=W_2 \qquad X_6=-W_3 \\ &X_7=X_8=0 \end{aligned} \tag{3.34}$$

where I is the unit 2×2 matrix and $\{W_1, W_2, W_3\}$ are the 2×2 matrices (2.24) that span $\mathrm{sl}_2(R)$.

4. Concluding remarks

In this paper we have explored the symmetry group of a nonlinear Burgers heat equation system initially introduced by Bluman and Cole [1] in their study of non-classical similarity solutions of the heat equation. As might be expected, the symmetry group of this system is similar to that of the heat equation in that the quotient algebra $\mathscr{B}=\mathscr{G}/\mathscr{A}\simeq \mathrm{sl}_2(R)$, where $\mathscr{A}$ is a maximal solvable ideal and $\mathscr{G}$ is the Lie algebra (compare e.g. Sastri and Dunn [10]).

An investigation of the prolongation space Lie algebra was used to provide an alternative derivation of the Bäcklund transformation to the equation system (1.2) and (1.3) which was obtained previously by Painlevé analysis. The Bäcklund transformation was obtained by forcing closure on the algebra at an appropriate level. The analysis also yielded a 2×2 linear matrix system (equations (3.16) and (3.17)) with integrability conditions (1.2) and (1.3).

It was also shown that the truncated seven dimensional Lie algebra $\mathscr{L}_1$ in the prolongation space possessed a maximal solvable ideal $\mathscr{A}_1$, with quotient algebra $\mathscr{B}_1=\mathscr{L}_1/\mathscr{A}_1\simeq \mathrm{sl}_2(R)$. This suggests that $\mathrm{sl}_2(R)$ should also play a major role in the open ended algebraic prolongation structure of which $\mathscr{L}_1$ is a truncated version.

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